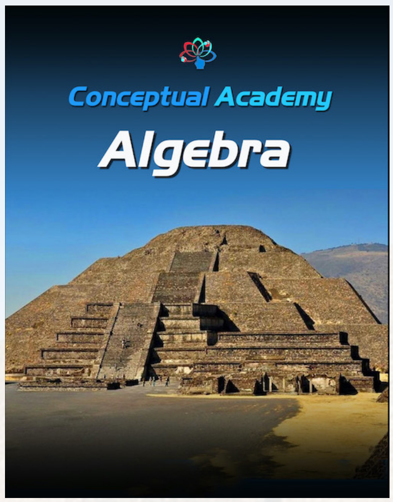
Conceptual Math
Algebra I

Chapter 2: Sets



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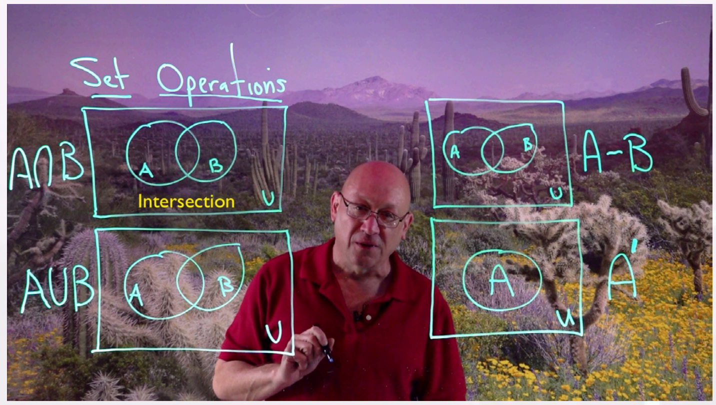

Chapter 2

Sets

2.1 Set Theory

Definition: A set is a collection of objects.

We usually name a set with a single capital letter (A, B, C, etc.).

Each object in the set is called an *element* of the set.

Any of three different notations are used to specify a set:

1. A *description* of the set using words / language. Example: Let C = the set of all cats currently available for adoption at the Cochise Animal Shelter.

2. A *list* or roster enclosed in brackets that specifies each object separated by commas. Example: Let the set B = { George, Tom, Dave, Cindy, Susan, Tina }.

3. *Set Builder Notation* which uses a variable to represent an element of the set and then provides a rule that the variable (element) must satisfy to be included in the set. A vertical bar "|" (interpreted as "such that") is placed between the variable and the rule. Example: Let G = {x | x is an even number between 3 and 11}. Set Builder Notation can also use mathematical formulas to specify a set, such as G = { x | 23 $< x <$ 45 }.

Note: A variable is a symbol (usually a lower-case letter) that is often used to represent an object. They are most often used to represent numbers in an equation or a formula, but they are also used to represent objects in a set. For example, let x = a person that likes chocolate more than vanilla. We will talk more about variables soon.

For many sets it is possible to use more than one notation to specify the same set. The following three are describing the same set:

The Set A = {2, 4, 6, 8}
The Set A = all even numbers between 1 and 9
The Set A = {x | x is even and $1 < x < 9$}

In set theory we use the Greek letter epsilon ∈ to mean "is an element of" when we wish to state that a particular object is contained in a set.

Dave ∈ A means that Dave is an element in the set A.

Definition: The *cardinal number* of a set is the number of elements in the set. When discussing sets we denote the cardinal number of a set A as n(A).

It is possible for a set to have no elements, a completely empty set. We call this the empty set or the null set. The most common symbol for this is ∅ , but in some cases you will see it shown as { }, braces with nothing between them.

Equality of Sets: We say that two sets are *equal* if they have the same elements.

Are the following two sets equal?

A = {3, 5, 7, 9, 11}   B = {x | x is an odd number between 2 and 12}

The elements in set A are clearly listed. Does B have the same elements? Yes. A and B are equal sets.

Speaking more precisely, A and B are equal sets if:

1. Every element that is in A is also in B.

2. Every element that is in B is also in A.

When listed, the order of the elements does not matter. If A = {2, 5, 4, 6} and B = {5, 4, 2, 6} the sets are equal because they have the same elements.

Also, if the list of the set contains an element more than once, this does not interfere with equality. If A has {1, 2, 3, 2, 4, 5, 2, 4} and B has {1, 2, 3, 4, 5} the two sets are equal. That A has multiple elements that are the same does not matter.

CAUTION: When discussing the equality of sets, be careful not to use the word "equivalent." In set theory the word "equivalence" is used to state that two sets have the same number of elements (the same cardinality).

Definition: The two sets A and B are *equivalent* if n(A) = n(B).

In some situations as we discuss sets, there is the concept of "everything," the set that contains everything in the universe of the objects under consideration. If the context were all human beings, it would be the set of all people. We call this the *universal set* and denote it as U.

U = the universal set that contains everything.

U does not refer to the entire universe. It is not everything. It is *everything under consideration* for our situation. If we were playing poker, we might consider U to be the deck of cards. If we were dining at a restaurant, U might be all of the items on the menu.

For a given set A, there are other sets related to A in certain ways.

Definition: The *complement* of a set A, denoted A', is the set of all elements that are not in A, or A' = {x | x $\in$ U and x $\notin$ A}.

To specify A' it is necessary to know the elements in the universal set U.

Example: Let the universal set U = every student at Sycamore High School and let the set A = girls on the track team.

What is A'? This will be every student at Sycamore High School who is not on the girl's track team.

Definition: B is a *subset* of A if every element in B is also an element of A.

Example : Let A = the set of all dogs and let B = the set of all golden retrievers. Every element in B (a golden retriever) is in the set of all dogs.

We denote that B is a subset of A by writing: $B \subseteq A$.

The definition of a subset makes it possible for it contain EVERY element of the original set, which is why you see the line underneath the subset symbol.

Since this is the case: For every set A, we have $A \subseteq A$.

What we usually mean when we talk about subsets are sets that contain some of the elements of the other set but not all of them. In the precise language of math, we specify these kinds of subsets as proper subsets.

Definition: B is a *proper* subset of A if $B \subset A$ and $A \neq B$.

We denote a proper subset by removing the line under the subset symbol, or $B \subset A$.

Food for Thought: How many different subsets are possible for a given set?

Say we have a set of three people: A = {Bo, Carrie, Shirley}

For simplicity, let's use the first letters of their names: {B, C, S}

How many subsets of A = {B, C, S} are possible?

Every set has the empty set as a subset: $\emptyset$

We have the original set itself: {B, C, S}

We have the three sets with one person: {B}, {C}, {S}

We have the three sets with two people: {B, C} {B, S} {C, S}

So if we add these up, we get 8 possible subsets for a set with three elements.

FACT: A set with n distinct elements has 2^n possible subsets.

If we insist on proper subsets, we lose the set itself (one set), so:

FACT: A set with n distinct elements has 2^n - 1 possible proper subsets.

2.2 Set Operations and Venn Diagrams

2.2.1 Set Operations

Several operations exist to combine sets.

INTERSECTION

Definition: The *intersection* of two sets, denoted A ∩ B, is the set of all elements that are contained in BOTH A and B.

$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$

If C = A ∩ B, then every element in C is contained in A, and every element in C is contained in B. Putting this into symbols, we can say that if C A ∩ B, then C ⊆ A and and C ⊆ B.

UNION

Definition: The *union* of two sets, denoted A ∪ B, is the set of all elements that contained in EITHER A or B.

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

Example: Let A = {1, 2, 3, 6, 9, 12, 15} and B = {2, 4, 5, 7, 9, 12}

Find: 1) $A \cap B$ and 2) $A \cup B$

1) To find $A \cap B$ we identify the elements that are contained in BOTH sets. Taking each number one at a time, we find that 2, 9, and 12 are in both. So $A \cap B = \{2, 9, 12\}$.

2) To find $A \cup B$, we list every number that is in either set. This gives us 1, 2, 3, 4, 5, 6, 7, 9, 12, 15. So we have $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9, 12, 15\}$.

SUBTRACTION

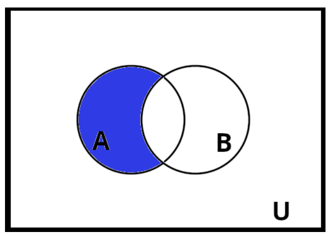

Figure 2.1: A - B

As we will learn in more detail later, the operation of subtraction produces what is called a difference.

Definition: The $\underline{difference}$ of two sets A and B, denoted $A - B$, is equal to the set $\{x \mid x \in A \text{ and } x \notin B\}$.

Saying this in words, we remove from the set A any elements that are in the set B. We are subtracting B from A.

Consider the two sets we had before:

$A = \{1, 2, 3, 6, 9, 12, 15\}$ $B = \{2, 4, 5, 7, 9, 12\}$

The elements in A that are also in B are 2, 9, and 12, so to get $A - B$ we remove these from A.

$A - B = \{1, 3, 6, 15\}$

2.2.2 Venn Diagrams

Figure 2.1 is an example of a Venn Diagram. Venn Diagrams are a useful way to illustrate operations with sets by providing a visual representation of set relationships and set operations.

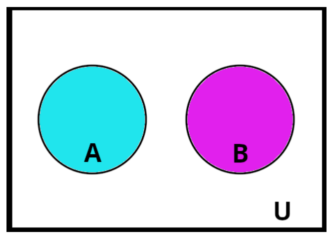

Figure 2.2: Venn Diagram

With Venn Diagrams we represent sets as simple shapes, usually circles or squares, but any closed shape (has a complete boundary) can work. We often represent the universal set as a large rectangle.

We usually specify sets and set operations by shading in or coloring portions or "spaces" that correspond to the set operations being performed. Note that for A ∩ B, the region contained in both A and B is shaded. For A ∪ B, everything in the two circles is shaded. A' (complement) shades everything (the universal set) that not inside (an element of) A.

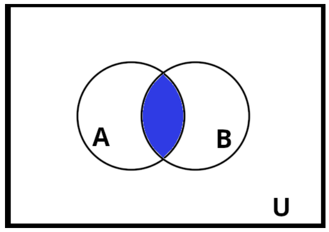

Figure 2.3: A intersect B.

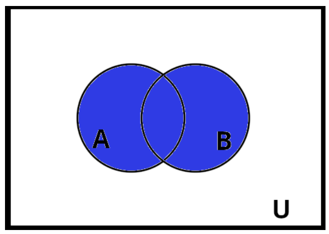

Figure 2.4: A Union B.

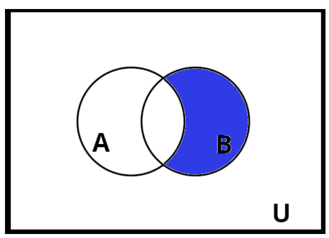

Figure 2.5: B - A

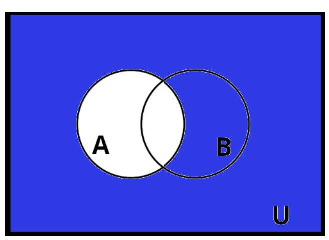

Figure 2.6: A Complement