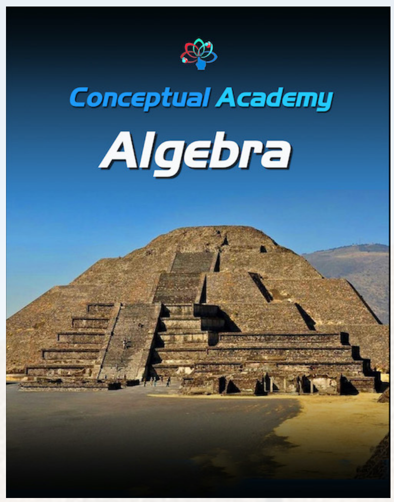
Conceptual Math
Algebra I

Chapter 3: Real Numbers

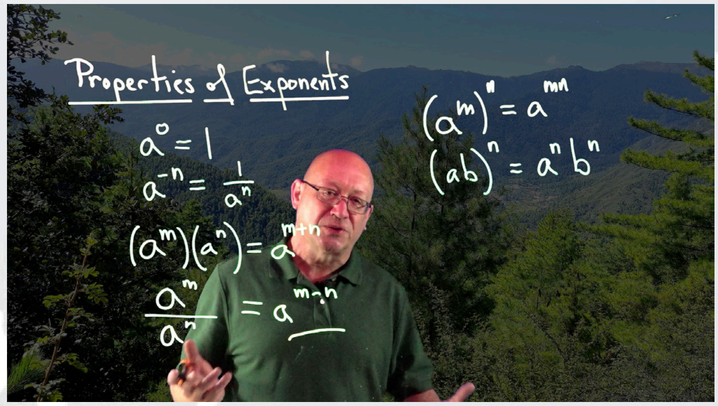


Matt Foraker, Ph.D.
Western Kentucky University
Bowling Green, KY

Conceptual Academy

All inquiries
Support@ConceptualAcademy.com

Chapter 3



Real Numbers

3.1 The Invention of Numbers

As early as human beings developed language, they acquired the ability to count. Counting was very important as we used it to measure how much food we had, how many tools we had, how many people were in our group.

As we gathered food, or wood for fire, or tools, and we added items into storage we counted up, we ADDED. As we ate food or burned the wood, we counted down, or SUBTRACTED.

We learned how to do this as soon as we learned how to talk and perhaps even before that. This gave us our first set of numbers, the Natural Numbers. The Natural Numbers consisted of the numbers we could see right in front of us as we counted up or down.

Natural Numbers = 0, 1, 2, 3, 4, 5, ... and they could get as big as we needed to count what we were counting.

Thousands of years passed.

We don't know exactly how the number to represent "nothing" was first invented. Different cultures approached the situation in their own ways, with the similar result - the number ZERO.

The inclusion of zero gives us the set of numbers we call the WHOLE Numbers.

Whole Numbers: 0, 1, 2, 3, 4, ...

As we grew more sophisticated we realized that in fact we could have a situation where a larger number was subtracted from a smaller number. Perhaps someone borrowed food or supplies with the promise that when possible it would be returned. In this case, we have less than zero because we have to pay back what we took. This introduced the negative numbers, and we have the number set known as the INTEGERS.

The Set of Integers: ...... − 4, − 3, − 2, −1, 0, 1, 2, 3, 4 ....

While we had been encountering the situation where the pieces of fruit or slices of meat did not match the number of people to feed, it took some time for us to create the math to handle this. We naturally knew that if we had two children and only one apple, we could cut the apple into equal pieces and give one to each kid.

To express this mathematically, we invented FRACTIONS. Fractions gave us a way to be very specific about dividing this up into portions. This produced what we call the RATIONAL Numbers.

RATIONAL NUMBERS: any number x where x can be expressed as $\frac{a}{b}$, where a and b are integers

The last number set to discuss the set of all numbers that are possible on a number line. This set is called the REAL numbers.

3.2 Real Numbers and The Number Line

For many years people believed that the rational numbers included every number, since fractions could be infinitely precise per our understanding at the time. While not certain, history suggests that the philosopher Hippasus, familiar with the Pythagoreans and the Pythagorean Theorem, starting thinking about right triangles.

The Pythagorean Theorem states that square of the hypotenuse of a right triangle is equal to the sum of the squares of the two adjacent sides. The was already a proven result in geometry. Hippasus considered a triangle whose two sides had length one. The theorem states that the hypotenuse must have length $\sqrt{2}$.

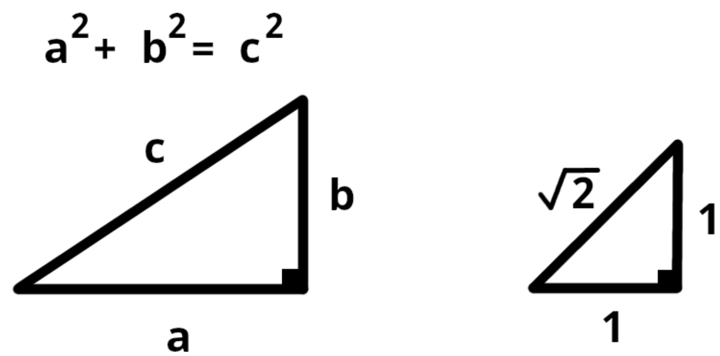


Figure 3.1: The Pythagorean Theorem

He tried to calculate this (no calculators back then). The number 1.41 was too small, but 1.42 was too large. Squaring 1.415 got very close to 2, but was a little too big. After weeks or months or years, he concluded that the decimals would never repeat, which MUST occur if it can be a fraction.

What number squared is 2?

1.414213562 and it kept going.

This created quite the kerfuffle with the Pythagoreans. What exactly happened is not clear, but while on a boat with his peers, Hippasus found himself no longer in the boat and drowned. Rumors spread that he perhaps had assistance leaving the boat.

Unfortunately for some, it was not possible to toss $\sqrt{2}$ overboard. Once understood, we recognized that there were many, in fact an infinite number of numbers that could not be expressed as fractions, i.e., not rational. We called them IRRATIONAL numbers. These numbers include π, e (the base of the natural logarithm), and many square roots, cube roots, and so on.

With both rational and irrational numbers, we captured every number possible on the number line. We call this set the REAL numbers.

FACT: Every point on the number line is a real number.

FACT: Every real number is a point on the line.

The Sets of Numbers:

Natural: 1, 2, 3, 4, 5, . . .

Whole: 0, 1, 2, 3, 4, 5, . . .

Integers: - 4, - 3, - 2, - 1, 0, 1, 2, 3, 4, . . .

Rational: Any number of the form $\frac{a}{b}$ where a and b are integers.

Real: Any number on the number line.

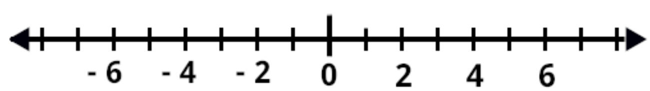


Figure 3.2: The Real Number Line

3.3 The Four Operations

The Four Operations

Addition: The $\underline{SUM}$ of two numbers a and b is denoted a + b. To compute the sum, we add a and b.

Subtraction: The $\underline{DIFFERENCE}$ of two numbers a and b is denoted a − b. To compute the difference, we subtract b from a.

Multiplication:The $\underline{PRODUCT}$ of two numbers a and b is denoted ab, a · b, (a)(b) or a*b (the star is usually only used when the other notations are not available, such as a computer program). To compute the product, we multiply a times b.

Division: The $\underline{QUOTIENT}$ of two numbers a and b is denoted a ÷ b, $\frac{a}{b}$, or a/b. Like the star * in multiplication, the slash / is usually used in computer programming. To compute the quotient, we divide a by b.

3.4 Properties of Real Numbers

The following hold for all real numbers. Memorization of the properties in terms of being able to list and state their names is NOT the required skill. What must occur is the repeated use of them until they are ingrained to be as natural as tying a shoe, brushing one's teeth, or seeing the time by looking at a clock.

Property	**Addition**	**Multiplication**
Commutative Property	a + b = b + a	ab = ba
Associative Property	(a + b) + c = a + (b + c)	(ab)c = a(bc)
Identities	a + 0 = a	(a)(1) = a
Inverses	a + (− a) = 0	(a) $\left(\frac{1}{a}\right) = 1$
Distributive Property	$a\ (b\ \pm\ c) = ab\ \pm\ ac$	

The Distributive Property is very important as we learn to work with numbers. When we are working with literal numbers whose value we know, it's possible to do the calculation using the commutative or associative properties, but when we introduce variables, which will happen very soon, the distributive law is essential.

For now, consider $4\ (3 + 2) = 4(5) = 20$

The distributive law tells us that $4\ (3 + 2) = (4)(3) + (4)\ (2) = 12 + 8 = 20$

Other examples:

Commutative: $3 + 5 = 5 + 3 = 8$

$(3)(5) = (5)(3) = 15$

Associative: $(6 + 8) + 7 = 14 + 7 = 21$

$6 + (8 + 7) = 6 + 15 = 21$

Identities: $5 + 0 = 5$ $(5)(1) = 5$

Inverses: $12 + (-\ 12) = 0$ $(7)\left(\frac{1}{7}\right) = 1$

3.5 Variables

A major distinction between algebra and arithmetic is algebra's inclusion of variables, numbers whose values are unknown or remain undetermined yet are still manipulated and used in calculations to solve problems or explore mathematical relationships.

Definition: A *literal number* is a number whose value is known. We express it by its value, 5, 3, – 7, $\sqrt{23}$.

Definition: A *variable* is a number whose value is not known. We express it with a letter. The best practice is to use lower case letters. For simplicity, we typically stay with x, y, and z, as well as a, b, and c, until the situation suggests other letters.

Examples: x, y, a, b.

Stick to lower case and NEVER use the letter o or l which look far too much like zero and one. The letter z is commonly used, but when writing by hand, ALWAYS cross the z as it is easy to interpret as the number 2.

Refrain from using i, j, or k as these have a special meaning that we will discuss later.

Note that we have already used variables (letters to represent numbers) when we introduced the properties of real numbers.

When we work with variables, it is important to understand that a literal number in front of a variable MEANS multiplication.

5x, (5)(x) and $5 \cdot x$ denote the multiplication of 5 times x.

We NEVER write a variable in front of a literal for this. NEVER: x5.

Two variables written next to each other implies multiplication.

xy, (x)(y), and $x \cdot y$ denote the multiplication of x times y.

3.6 Exponents

An exponent is notation to indicate a number being multiplied by itself a given number of times.

By definition,

$a^1 = a$

$a^2 = a \cdot a$

$a^3 = a \cdot a \cdot a$

$a^4 = a \cdot a \cdot a \cdot a$

$a^n = a \cdot a \cdot a \cdot a \ldots a$ (n times)

3.6.1 Properties of Exponents

For any real number a, we have:

$a^0 = 1.$

$a^{-n} = \frac{1}{a^n}$

$(a^{m}) \cdot (a^{n}) = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Like the properties of real numbers, the desired skill is a working knowledge developed by repeated use in simplifying expressions that contain exponents.

Examples:

Simplify $x^3 \cdot x^4$

We have $x^3 \cdot x^4 = x^{3+4} = x^7$.

Simplify: $\frac{x^8}{x^5}$

$\frac{x^8}{x^5} = x^{8-5} = x^3$

Simplify $(x^3)^4$

$(x^3)^4 = x^{(3)(4)} = x^{12}$

Simplify $(x^3y^4z^2)^5$

$(x^3y^4z^2)^5 = (x^3)^5(y^4)^5(z^2)^5$

which is $(x^{15})(y^{20})(z^{10})$

Or $x^{15}y^{20}z^{10}$

Simplify $\frac{x^3y^7}{x^6y^5}$

We get $x^{3-6}y^{7-5} = x^{-3}y^2 = \frac{y^2}{x^3}$

3.7 Roots, Radicals and Rational Exponents

Suppose we want to ask the question, ”What number squared is 225?” or ”What number raised to the 4th power is 81?” These are called roots. Like exponents, we use ”square” when the power is two and ”cube” when the power is three. Roots are commonly denoted using the radical sign $\sqrt{}$. When no number occurs above and before the sign, it is a square root. For higher roots, a number is placed in front of the symbol.

The square root of a, denoted $\sqrt{a}$, is the number which when squared gives us a. By definition, $(\sqrt{a})^2 = a$.

The cube root of a, denoted $\sqrt[3]{a}$, is the number which when cubed gives us a. By definition, $(\sqrt[3]{a})^3 = a$.

The 4th root of a, denoted $\sqrt[4]{a}$, is the number which when raised to the fourth power gives us a. By definition, $(\sqrt[4]{a})^4 = a$.

DEFINITION: c is called the nth root of a if $c^n = a$. $c = \sqrt[n]{a}$

When n is even, the sign of a root can be positive or negative. For example, both 2^2 and $(-2)^2$ are 4, so one can say that both are the square root of 4. We refer to the positive value as the *principle* root. When asked to find roots, we typically answer with the principle root only.

Since radicals are exponents, the properties of exponents apply to roots, so we have:

Properties of Radicals

$\sqrt[n]{a^n} = |\, a \,|$ when n is even. We take the principle (positive) root.

$\sqrt[n]{a^n} = a$ when n is odd.

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[n]{a^m} = (\sqrt[n]{a})^m$

3.7.1 Working with Radicals

A common task associated with radical expressions is to simply them. Simplifying radical expressions consists of: 1) removing roots that can be taken from inside the radical, 2) eliminating the radical signs found in the denominators of fractions, and 3) applying properties of radicals as warranted to simplify the expression.

EXAMPLES:

Simplify $\sqrt{48}$

Solution: We know that $\sqrt{48} = \sqrt{16 \cdot 3}$

which is $\sqrt{16} \cdot \sqrt{3} = 4 \cdot \sqrt{3}$

The expression cannot be simplified further.

Simplify $\sqrt{72}$

Solution: We know that $\sqrt{72} = \sqrt{9 \cdot 8}$

which is $\sqrt{9} \cdot \sqrt{8} = 3 \cdot \sqrt{8}$

but this is $3 \cdot \sqrt{4 \cdot 2} = 3 \cdot \sqrt{4} \cdot \sqrt{2}$

or $3 \cdot 2 \sqrt{2}$

leaving us with $6 \sqrt{2}$.

Simplify $\sqrt{150}$

Solution: We know that $\sqrt{150} = \sqrt{25 \cdot 6}$

which is $\sqrt{25} \cdot \sqrt{6} = 5 \cdot \sqrt{6}$

Simplify $\sqrt{x^3 y^4}$

Solution: We know that $\sqrt{x^3 y^4} = \sqrt{x^3 \cdot y^4}$

which is $\sqrt{x^3} \cdot \sqrt{y^4} = y^2 \cdot \sqrt{x^2 \cdot x} = y^2 \cdot \sqrt{x^2} \cdot \sqrt{x}$

or $xy^2\sqrt{x}$.

Simplify $\sqrt{36x^8y^5}$

Solution: We know that $\sqrt{36x^8y^5} = \sqrt{36 \cdot x^3 \cdot y^4}$

which is $\sqrt{36} \cdot \sqrt{x^8} \cdot \sqrt{y^5} = 6x^4 \cdot \sqrt{y^5} = 6x^4y^2\sqrt{y}$

Simplify $\sqrt[3]{81x^6y^8}$

$\sqrt[3]{81x^6y^8} = \sqrt[3]{81} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{y^8}$

Or $\sqrt[3]{27 \cdot 3} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{y^6 \cdot y^2}$

Which is $3 \cdot x^2 \cdot y^2 \sqrt[3]{3y^2}$

Or $3x^2y^2\sqrt[3]{3y^2}$

Simplify $\sqrt[5]{x^{10}}$

$\sqrt[5]{x^{10}} = \sqrt[5]{x^5 \cdot x^5}$

$\sqrt[5]{x^5} \cdot \sqrt[5]{x^5}$

$x \cdot x$

x^2

When radicals appear in fractions or contain fractions, removing the radical signs from the denominator is considered a simplification. This process is called "rationalizing the denominator."

Simplify: $\dfrac{1}{\sqrt{2}}$

Multiple both numerator and denominator by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$$

Simplify: $\dfrac{3}{\sqrt{5}}$

Multiple both numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \frac{3\sqrt{5}}{5}$$

Simplify: $\dfrac{x}{\sqrt{y}}$

Multiple both numerator and denominator by $\sqrt{y}$.

$$\frac{x}{\sqrt{y}}\left(\frac{\sqrt{y}}{\sqrt{y}}\right) = \frac{x\sqrt{y}}{y}$$

3.7.2 Rational Exponents

Remember that radicals ARE exponents denoted with a radical sign. One could eliminate the use of the radical sign by using rational (think fractions) exponents.

$\sqrt{a} = \mathrm{a}^{\frac{1}{2}}$

$\sqrt[3]{a} = \mathrm{a}^{\frac{1}{3}}$

$\sqrt[4]{a} = \mathrm{a}^{\frac{1}{4}}$

$\sqrt[n]{a} = \mathrm{a}^{\frac{1}{n}}$

Notice that:

$\sqrt{a} \cdot \sqrt{a} = \mathrm{a}^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = \mathrm{a}^{\frac{1}{2}+\frac{1}{2}} = \mathrm{a}^1 = \mathrm{a}$

$\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} = \mathrm{a}^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = \mathrm{a}^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = \mathrm{a}^1 = \mathrm{a}.$

We also have:

$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{\frac{m}{n}}$

Since rational exponents are exponents, the properties of exponents apply. Converting radical signs to rational exponents allows one to simplify expressions more easily than can be done with the radical symbols.

With radical notation, even a rather simple expression can appear confusing.

Consider: $\sqrt[3]{a} \cdot \sqrt{a} \cdot \sqrt[5]{a}$

Expressed with rational exponents the same expression becomes trivial.

$a^{\frac{1}{3}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{5}} = \mathrm{a}^{\frac{1}{3}+\frac{1}{2}+\frac{1}{5}} = \mathrm{a}^{\frac{21}{30}} = a^{\frac{7}{10}}$ which we could write as $\sqrt[10]{a^7}$.

Example:

Simplify: $\sqrt{\sqrt[3]{a^2} \cdot \sqrt[5]{a}}$

Before panicking, simply write the expression with rational exponents to get:

$(a^{\frac{2}{3}} \cdot a^{\frac{1}{5}})^{\frac{1}{2}} = (a^{\frac{2}{3}+\frac{1}{5}})^{\frac{1}{2}} = (a^{\frac{13}{15}})^{\frac{1}{2}} = a^{\frac{13}{30}}$

Almost nobody likes to do this.