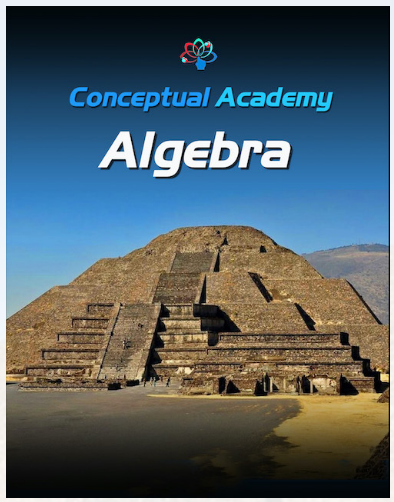
Conceptual Math
Algebra I

Chapter 13: Complex Numbers



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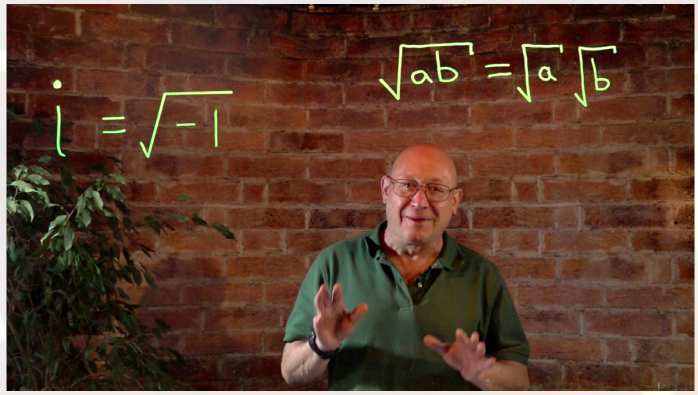

Chapter 13

Complex Numbers

13.1 Complex Numbers

We have to this point studied numbers on the real number line. Just as Hippassus noticed that no rational number could represent $\sqrt{2}$, we have the situation where no REAL number can represent $\sqrt{-1}$ or in fact the square root of any number less than zero. For any real number x, we have $x^2 > 0$.

We address this by setting the variable i to represent the value $\sqrt{-1}$. This is NOT a real number. It has no place on the real number line. To distinguish i and all multiples of i we refer to these as *imaginary numbers*.

We designate $i = \sqrt{-1}$.

By definition, this means that $i^2 = -1$.

The properties of roots tell us that $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Consider $\sqrt{-7}$

$\sqrt{(-1)(7)}$

$\sqrt{(-1)}\sqrt{7}$

$i\sqrt{7}$

The square root of any negative number can be expressed as i multiplied by the square root of a positive number.

For any real number $a > 0, \sqrt{-a} = i\sqrt{a}$

Again, i and any multiple of i are called IMAGINARY numbers. *This does not mean they don't exist!* They are NOT REAL in the context of being a real number. They exist as much as any other number.

Imaginary numbers: 3i, 7i, –4i, 220i, i$\sqrt{13}$.

PROTOCOL: If the multiple of i is a root, we write the i before the root. If we have $\sqrt{5}$ times i, we write $i\sqrt{5}$.

Real and imaginary numbers cannot be added or subtracted to produce a single term. When we have a sum or difference of real and imaginary numbers, we express them as a + bi where a and b are real numbers (both a and b can be negative).

A COMPLEX NUMBER is a number of the form a + bi, where a and b are real numbers.

The number a is referred to as the real part of the number.
The number b is referred to as the imaginary part of the number.

Complex Numbers: $3 + 7i, 5 - 2i, 12 + i\sqrt{3}$

While it is not incorrect to consider a + bi the sum of two numbers, it is more accurate and mathematically sound to understand that a + bi is in fact ONE NUMBER.

13.2 Operations with Complex Numbers

The Four Operations with Complex Numbers

Addition and Subtraction

Adding and subtracting complex numbers is essentially the same as combining like terms when we add or subtract polynomials. We combine the real parts and combine the imaginary parts.

Add (6 + 4i) + (3 + 2i)

We get (6 + 3) + (4 + 2)i

Or 9 + 6i

Subtract (3 + 7i) – (1 + 4i)

So (3 – 1) + (7 – 4)i

Or 2 – 3i

Multiplication

To Multiply – We multiply using FOIL just as we did with binomials.

Multiply (3 + 2i)(5 + i)

We get (3)(5) + (3)(i) + (2i)(5) + (2i)(i)

Or $15 + 3i + 10i + 2i^2 2$

Note that i squared is negative one.

So we get 15 + 13i – 2 = 13 + 13i

Multiply (4 – 3i)(5 + 2i)

(4)(5)+(4)(2i)+(– 3i)(5)+ – 3i)(2i)

$20 + 8i–15i–6i^2$

Note that the last term becomes positive because of the squared i.
26 – 7i.

Division

Divide (2 + 3i) / (4 + 2i)

To divide complex numbers, we use what is called the complex conjugate.

Definition: The complex conjugate of the complex number a + bi is a – bi.

The complex conjugate of 6 + 3i is 6 – 3i.

The complex conjugate of 7 – 4i is 7 + 4i.

To divide two complex numbers, express the quotient as a fraction. Multiply both the numerator and denominator by the complex conjugate *of the denominator*. This will result in a real number in the denominator. Use FOIL to simplify the numerator.

Divide $\frac{2+3i}{4+2i}$

$$\frac{2+3i}{4+2i} \cdot \frac{4-2i}{4-2i}$$

$$\frac{(2+3i()4-2i)}{(4+2i)(4-2i)}$$

Apply FOIL to both numerator and denominator.

$$\frac{8-4i+12i-6i^2}{16-8i+8i-4i^2}$$

$$\frac{8+8i+6}{16+4}$$

$$\frac{14+8i}{20}$$

$$\frac{7+4i}{10}$$

which could also be written as $\frac{7}{10}+\frac{2}{5}i$.

Divide $\quad \frac{5-3i}{2-7i}$

$$\frac{5-3i}{2-7i} \cdot \frac{2+7i}{2+7i}$$

$$\frac{(5-3i)(2+7i)}{(2-7i)(2+7i)}$$

$$\frac{15+35i-6i-21i^2}{4+14i+14i-49i^2}$$

$$\frac{15+29i+21}{4+49}$$

$$\frac{36+29i}{53}$$

$$\frac{36}{53}+\frac{29}{53}i$$

Notice that the product of a complex number and its complex conjugate always eliminates the imaginary parts and produces a real number.

USEFUL FACT: For any complex number a + bi:

(a + bi)(a - bi) = $a^2 + b^2$.