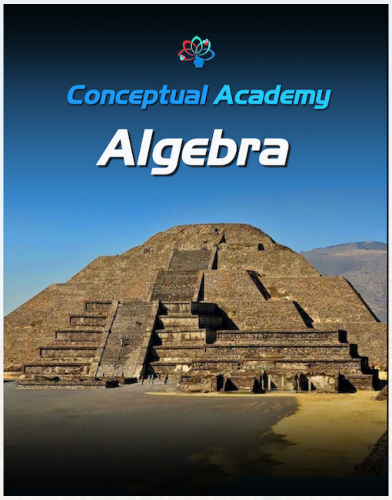
Conceptual Math
Algebra I

Chapter 14: Functions



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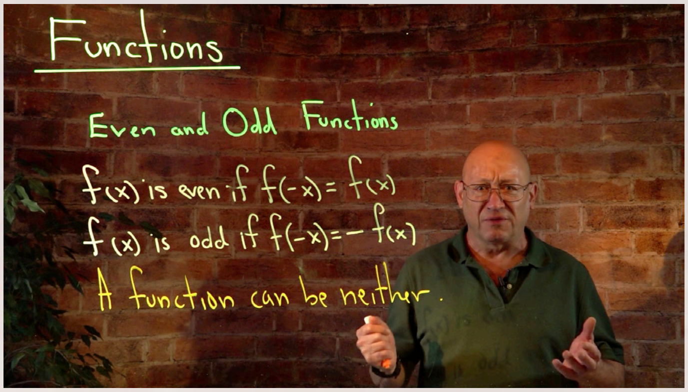

Chapter 14

Functions

14.1 Functions

By setting two ezpressions equal to each other, an equation is describing a relationship that exists between the quantities expressed. The equation $y = 2x$ tells us that the value of y is twice the value of x. If we were to plot points and graph the solution to the equation $x^2 + y^2 = 4$, we would see that it is a circle centered at the origin with a radius of 2.

When the relationship between the variables meets a certain criteria, we refer to them with various names.

A relation between two variables x and y is a set of ordered pairs (x,y). The set can be infinite. Relations have the property that for every value of x in the relation, there is at last one corresponding value for y, and vice versa.

When we think of the first number x as an "input" and the second number y as an "output," then if the relation also has the property that for every value of x there is one AND ONLY ONE value of y, we call that relation a function and we write y = f(x).

The equation y = 2x is both a relation and a function. For every value of x there is one and only one value of y. The equation $x^2 + y^2 = 4$ is a relation, since for every value of x in the equation there is a corresponding value of y, but it is NOT a function. For the value x = 0 we can have y = 2 or y = -2.

The function y = f(x) can be thought of as a formula that calculates a value of y for every value of x for which the function is defined. (For example, y = f(x) = $\frac{x}{y}$) is a function, but it is not defined for x = 0 since we cannot divide by zero.)

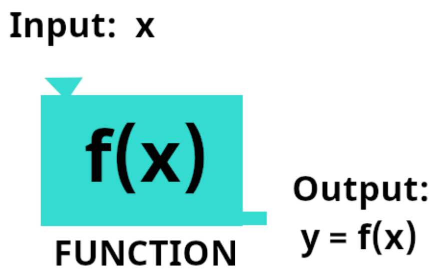

Figure 14.1: The Function

When we graph an equation in x and y, the "vertical line test" can determine if the equation defines y as a function of x. To perform the vertical line test, one sweeps a vertical line across all values of x and examines the graph. Does the graph ever intersect the line more than once? If so, the equation is NOT a function.

IMAGE - as the vertical line is moved back and forth, does it ever intersect the graph more than once?

Consider y = 2x + 3.

Consider $x^2 + y^2 = 4$

For f(x) = 3x - 7 evaluate f(5), f(-3), and f(2x).

f(5) = 3(5) - 7 = 15 - 7 = 8.

f(-3) = 3(-3) - 7 = - 9 - 7 = - 16.

f(2x) = 3(2x) - 7 = 6x - 7.

For f(x) = $x^2 + 5$ evaluate f(2), f(-3), and f(3x).

f(2) = $2^2 + 5$ = 4 + 5 = 9.

f(-3) = $(-3)^2 + 5$ = 9 + 5 = 14.

f(3x) = $(3x)^2 + 5 = 9x^2 + 5$.

For f(x) = $\frac{3}{x-2}$ evaluate f(3), f(0), and f(x+5).

f(3) = $\frac{3}{3-2} = \frac{3}{1} = 3$.

f(0) = $\frac{3}{0-2} = \frac{3}{-2} = -\frac{3}{2}$.

f(x+5) = $\frac{3}{(x+5)-2} = \frac{3}{x+3}$.

For f(x) = $(x+2)^2$ evaluate f(2), f(3a), and f(x - 1).

f(2) = $(2+2)^2 = 4^2 = 16$.

f(3a) = $(3a+2)^2 = 9a^2 + 12a + 4$.

f(x - 1) = $((x-1)+2)^2 = (x+1)^2 = x^2 + 2x + 1$.

CRITICAL SKILLS:

1. Master function notation including function algebra, composition, inverses of functions.
2. Evaluate functions for a given input including expressions
3. Function algebra
4. Function composition.
5. Inverses of functions.

14.2 Key Functions

Algebra students should become very clear about seven key functions that frequently arise in various situations. They are:

1. The constant function f(x) = k
2. The identity function f(x) = x
3. The linear function f(x) = $mx + b$
4. The squaring function f(x) = x^2
5. The absolute value function f(x) = $| x |$
6. The square root function f(x) = $\sqrt{x}$
7. The reciprocal function f(x) = $\frac{1}{x}$

14.3 Function Algebra

One can perform the four operations on functions. We use the following notation.

Addition: f+g(x) = f(x) + g(x)

Subtraction: f-g(x) = f(x) - g(x)

Multiplication: fg(x) = f(x) · g(x)

Division: $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$

Find f+g(x), f-g(x), fg(x), and $\frac{f}{g}(x)$ for the following.

f(x) = 2x + 5 and g(x) = 3x - 7.

f+g(x) = f(x) + g(x) = (2x + 5) + (3x - 7) = 5x - 2

f-g(x) = f(x) - g(x) = (2x + 5) - (3x - 7) = - x + 12

fg(x) = f(x) · g(x) = (2x + 5) · (3x - 7) = 6x^2 + x - 35

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{2x+5}{3x-7}$$

f(x) = x + 2 and g(x) = x^2.

f+g(x) = f(x) + g(x) = (x + 2) + x^2 = x^2 + x + 2

f-g(x) = f(x) - g(x) = (x + 2) - x^2 = - x^2 + x + 2

fg(x) = f(x) · g(x) = (x + 2) · x^2 = x^3 + 2x^2

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x+2}{x^2}$$

f(x) = $\frac{5}{x}$ and g(x) = 4x.

f+g(x) = f(x) + g(x) = $\frac{5}{x}$+ 4x = x^2 + x + 2

f-g(x) = f(x) - g(x) = $\frac{5}{x}$ - 4x = - x^2 + x + 2

fg(x) = f(x) · g(x) = $\frac{5}{x}$ · 4x = 20

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\frac{5}{x}}{4x} = \frac{5}{4x^2}$$

f(x) = x^3 and g(x) = x^2.

f+g(x) = f(x) + g(x) = x^3+ x^2 = x^3 + x^2

f-g(x) = f(x) - g(x) = x^3 - x^2 = x^3 - x^2

fg(x) = f(x) · g(x) = $x^3 \cdot x^2 = x^5$

$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^3}{x^2}$ = x.

14.4 Function Composition

The situation where a number is put into a series of functions is common in practical applications in many fields. When this occurs, we say that we have a composition of functions.

Given two functions f(x) and g(x), we define the following:

The composition of functions f and g is denoted f o g (x). We calculate f o g (x) = f(g(x)). Notice that the language designates the order. The composition of functions g and f is denoted g o f (x). We calculate g o f (x) = g(f(x))

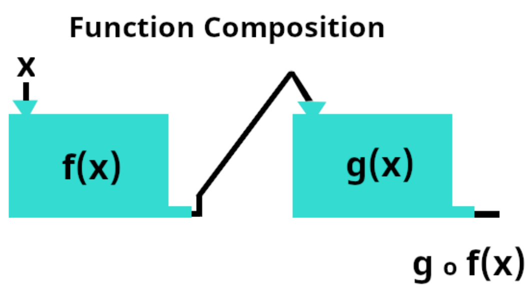

Figure 14.2: Function Composition

Example:

For f(x) = 2x + 3 and g(x) = x^2 find:

f o g (x)

g o f (x)

f o g (3)

We have:

f o g (x) = f(g(x)) = 2 (x^2) + 3 = 2x^2 + 3

g o f (x) = g(f(x)) = $(2x+3)^2$ = (2x + 3)(2x + 3) = 4x^2+ 12x + 9

f o g (3) = $2(3)^2$ + 3 = 18 + 3 = 21

14.5 Inverses of Functions

The inverse of a function is another function that performs the opposite operation. Put another way, it is the "reverse" of the original function, restoring the input back to what it was before the initial function was performed. The inverse of adding five is subtracting five.

We denote the inverse of f(x) as $f^{-1}(x)$. Note that the superscript is NOT an exponent! If f(x) = x + 5, then $f^{-1}(x)$ = x - 5. Notice that if we perform a function and then its inverse, we are back where we started.

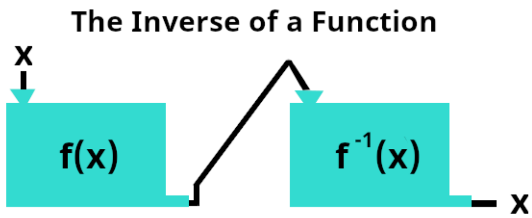

Figure 14.3: Inverse of a Function

Examples:

If f(x) = x - 12, then $f^{-1}(x)$. = x + 12

If f(x) = 7x, then $f^{-1}(x) = \frac{x}{7}$

If f(x) = $\frac{x}{15}$, then $f^{-1}(x)$ = 15x

How to find the inverse f^{-1}(x) for a given function f(x):

1. Write the equation y = f(x) with the formula for f(x).

2. Switch every x and y in the equation.

3. Solve for y in the new equation.

4. The solution is f^{-1}(x).

Example: Find f^{-1}(x) for f(x) = 3x + 4.

1. Write y = 3x + 4

2. Interchange x and y to get x = 3y + 4

3. Solve for y. 3y = x − 4

4. f^{-1}(x) = $\frac{x-4}{3}$

Example: Find f -1(x) for f(x) = (7x – 5)/4

1. Write y = (7x – 5) / 4

2. Interchange x and y to get x = (7y – 5)/4

3. Solve for y. 4x = 7y – 5

4. f^{-1}(x) = 4x + 5 = 7y

@ f^{-1}(x) = (4x + 5)/7

For the above two problems, find f o f^{-1}(x).
In the first, we have f o f^{-1}(x) = f(f^{-1}(x)) = f((x – 4)/3)

= 3 ((x – 4)/3) + 4

= (x – 4) + 4

= x

In the first, we have f o f^{-1}(x) = f(f^{-1}(x)) = f((4x + 5)/7)

$$= \frac{7\ (\frac{4x+5}{7}) - 5)}{4}$$

$$= \frac{(4x-5)+5}{4}$$

$$= \frac{4x}{4}$$

= x

Fact: For any function f with a defined inverse f^{-1}(x), we have:

f o f^{-1}(x) = f^{-1}(x) o f (x) = x

14.6 Even and Odd Functions

Definition: A function f(x) is <u>even</u> if f(-x) = f(x) for all x in the domain of f(x). A function f(x) is <u>odd</u> if f(-x) = -f(x) for all x in the domain of f(x).

NOTE:
1. The concept of even and odd for functions is DIFFERENT than the meaning they have for numbers.
2. A function can be even, odd, or neither. Most functions are neither.

To determine if a function is even or odd.

1. Replace the variable x with -x in the formula for f(x).
2. Simplify the formula.
3. If the result is equal to f(x), the function is even.
4. If the result is equal to -f(x), the function is odd.
5. If the result is not one of the two above, it is neither even nor odd.

EXAMPLES:

Determine if f(x) = $3x + 5$ is even or odd.
1. f(-x) = $(-x)^2 + 5$
2. Simplifies to f(x) = $x^2 + 5$
3. This is the same as f(x). The function is even.

Determine if f(x) = 7x - 2 is even or odd.

1. f(-x) = 7(-x) - 2
2. Simplifies to f(-x) = -7x - 2
3. Not the same as f(x). Not even.
4. -f(x) = -(7x - 2) = -7x + 2. Not odd.
5. The function is neither even nor odd.

Determine if f(x) = $x^3 + 5x$ is even or odd.

1. f(-x) = $(-x)^3 + 5(-x)$
2. Simplifies to f(-x) = $-x^3 - 5x$
3. Not the same as f(x). Not even.
4. -f(x) = $-x^3 - 5x$. The function is odd.