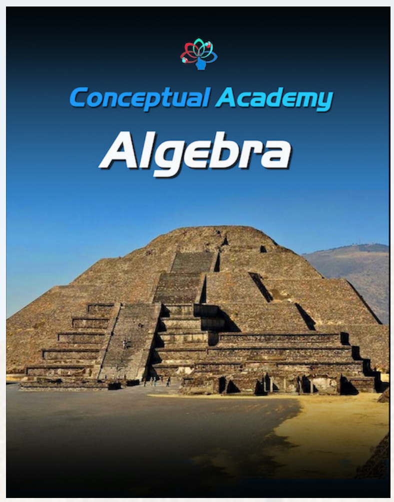
Conceptual Math
Algebra I

Chapter 18: Polynomial Equations and Graphs



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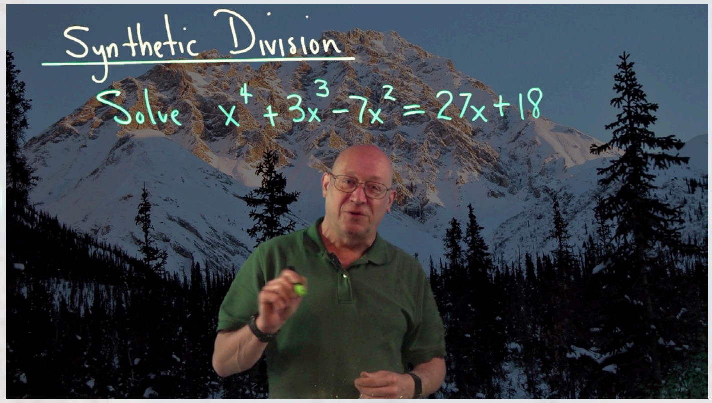

Chapter 18

Polynomial Equations and Graphs

18.1 The Principle of Zero Products

To solve equations involving polynomials, like quadratic equations, involves subtracting what is necessary to one side of the equation equal to zero. We then factor the polynomial and use the principle of zero products to identiy the solutions.

As noted earlier the principle of zero products states that if a product of numbers results in the value of zero, then at least one of those numbers MUST be zero.

IF AB = 0, the A = 0 or B=0 or both are zero. This extends to a product of three factors, four factors, and so on. If ABC = 0, then A=0 or B=0 or C=0 provide solutions to the equation.

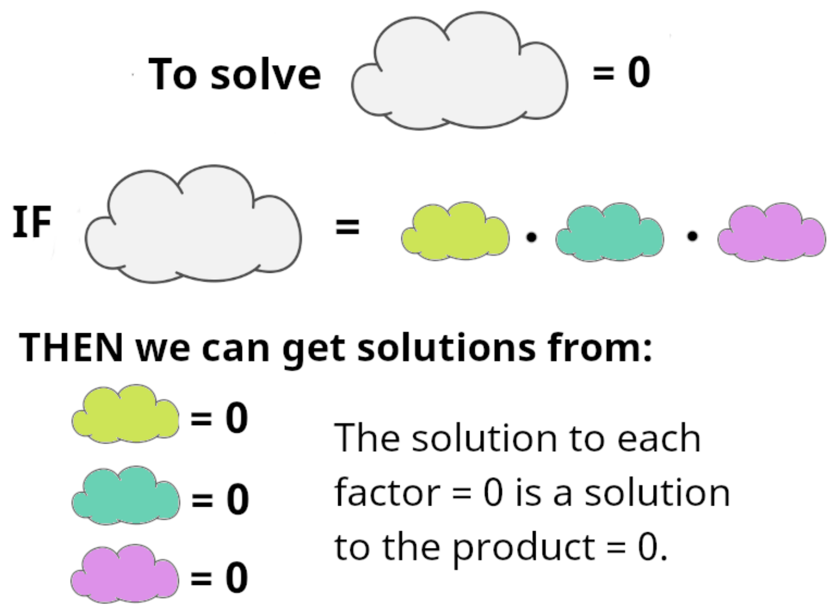

Figure 18.1: The Principle of Zero Products

Given a polynomial p(x), the values of x that satisfy the equation p(x) = 0 are called the ZEROS or the ROOTS of the polynomial. Here the word ROOT is referring to the zeros of a function, not a radical. The way to do find the zeros is to factor the polynomial completely, and then setting each of the factors to zero provides the solutions to the equation, which we are the zeros by definition.

Example: $x^2 – 5x = 6.$

Set to zero: $x^2 – 5x = 6 = 0$

Factor $(x + 1)(x – 6) = 0$

Principle of zero products: x = 6, – 1.

The zeros are 6 and – 1.

With quadratics, of course, we have the quadratic formula to find the zeros, but with higher order polynomials, we have no choice but to factor it.

Example: Find the zeros of $p(x) = x^3 + 2x^2 – 5x + 6$

Solution: Set to zero. $p(x) = x^3 + 2x^2 – 5x + 6 = 0$

Factor: $(x–1)(x + 2)(x–3) = 0$

Principle of zero products x = 1, – 2, and 3.

The zeros are 1, – 2, and 3.

Example Find the zeros of $f(x) = x^4 + 2x^3 – 7x^2 – 8x + 12$

Set to zero $f(x) = x^4 + 2x^3 – 7x^2 – 8x + 12 = 0$

Factor $(x + 2)(x–1)(x + 3)(x–2)$

Principle of zero products x = 1, 2, – 2 and – 3.

The zeros are 1, 2, – 2 and – 3.

Multiplicity: It is possible for a polynomial to factor in such a way that some of the factors occur more than once.

Consider $p(x) = (x–2)^3(x + 5)(x–3)$.

This polynomial has the zeros of x = 2, – 5, and 3. But the (x – 2) is cubed. When this happens, we use the term multiplicity. Here, we would say that the zero 2 has multiplicity 3. The other zeros have multiplicity 1.

18.2 The Fundamental Theorem of Algebra

One of the most brilliant mathematicians in the history of humanity, Karl Friedrich Gauss, produced one of the most famous theorems in all of Mathematics, the Fundamental Theorem of Algebra.

The Fundamental Theorem of Algebra: A polynomial of degree n has n zeroes when we include multiplicity. Granted, some of these roots might be complex numbers.

Said another way, it is possible to express every polynomial p(x) as a product of binomials. Note that these may have complex numbers.

can be written as: $p(x) = a_n(x–c_1)(x–c_2)(x–c_3) -----------(x–c_n)$
The c_i might be complex.

18.2.1 Synthetic Division

The Fundamental Theorem of Algebra tells us that a polynomial can be factored into the form:

$p(x) = a_n(x–c_1)(x–c_2)(x–c_3) -----------(x–c_n)$.

To do this we use an algorithm for dividing a polynomial p(x) by the binomial x – k. Recall that if we divide A by b and don't have a remainder (the remainder is zero) we say that A is divisable by b. This also means that if we factor A, b will be one of the factors.

Similarly, if we divide a polynomial p(x) by a binomial x - k and get a remainder of zero, we know that x - k is a factor of p(x) and that k is a zero of p(x).

To be clear:

What we have: a polynomial p(x) of any order ¿ 2.

What we want: the polynomial in factored form.

How we do it:

1. We test possible zeros using division algorithm called synthetic division.

2. Each zero we find removes a factor and leaves us with a smaller polynomial.

3. Repeat step two until all factors are found.

Consider the polynomial p(x) = $x^3 + 3x^2 – 10x + 24$

Since the constant term of p(x) is 24, our candidates for k are the numbers that can multiply to 24, and note that they can be of either sign, so our list consists of positive and negative 1, 2, 3, 4, 6, 8, 12, 24.

x-1 (x=1)	1	– 3	– 10	24	
		1	– 2	– 12	
	1	– 2	– 12	12	NO, x-1 is NOT a factor.

Now try x+1 (the zero x = -1).

x+1 (x= - 1)	1	– 3	– 10	24	
		– 1	4	6	
	1	– 4	– 6	30	NO, x+1 is NOT a factor.

Now try x-2 (the zero x = 2).

x+1 (x= 2)	1	– 3	– 10	24	
		2	– 2	– 24	
	1	– 1	– 12	0	YES! x-2 IS a factor.

This tells us that x = 2 is a zero and that (x – 2) is a factor.

It also tells us that $x^3 – 3x^2 – 10x + 24 = (x – 2)(x^2 – x – 12)$.

We know how to factor trinomials to get that $x^2 – x – 12$ = (x – 4)(x + 3)

We get p(x) = (x – 2)(x – 4)(x + 3). The zeros are 2, 4, and – 3.

Find the zeros of p(x) = $x^3 + 3x^2 – x – 3$

Our candidates are positive and negative 1 and 3.

Starting with 1.

x-1 (x=1)	1	3	– 1	- 3	
		1	4	3	
	1	4	3	0	YES, x - 1 is a factor.

We have p(x) = $x^3 + 3x^2 – x – 3$ = (x – 1)(x^2 + 4x + 3)

= (x – 1)(x + 1)(x + 3). The zeros are 1, – 1, and – 3.

Find the zeros of p(x) = $x^3 – 3x^2 – 4x + 12$

Our candidates are the factors of 12. We have plus or minus 1, 2, 3, 4, 6, 12.

Starting with 1.

x-1 (x= 1)	1	– 3	– 4	12	
		1	– 2	6	
	1	– 2	– 6	6	NO. x - 1 is not a factor.

Trying x + 1

x+1 (x= -1)	1	– 3	– 4	12	
		– 1	4	0	
	1	– 4	0	12	NO. x + 1 is not a factor.

Now try x – 2:

x-1 (x= 2)	1	– 3	– 4	12	
		2	– 2	12	
	1	– 1	– 6	0	YES. x - 2 is a factor.

We have $x^3 – 3x^2 – 4x + 12 = (x – 2)(x^2 – x – 6)$

= (x – 2)(x + 2)(x – 3). The zeros are 2, – 2, and 3.

Sometimes it comes to us as an equation.

Solve $x^4 + 3x^3 – 7x^2 = 27x + 18$

Set = 0: $x^4 + 3x^3 – 7x^2 - 27x - 18 = 0$

Since the last term is 18, our candidates are plus/minus 1, 2, 3, 9, 18.

Start with 1.

x-1 (x= 1)

$$\begin{array}{rrrrr} 1 & 3 & -7 & -27 & 18 \\ & 1 & 4 & -3 & -30 \\ \hline 1 & 4 & -3 & -30 & -12 \end{array} \quad \text{NO.}$$

Now try x = -1.

x+1 (x= -1)

$$\begin{array}{rrrrr} 1 & 3 & -7 & -27 & -18 \\ & -1 & -2 & 9 & 18 \\ \hline 1 & 2 & -9 & -18 & 0 \end{array} \quad \text{YES.}$$

So we have:

p(x) = $x^4 + 3x^3 – 7x^2 – 27x – 18 = (x + 1)(x^3 + 2x^2 – 9x – 18)$

Now we must factor the remaining cubic. Since 1 was not a zero of the original, it cannot be a zero of remaining cubic.

Move to x = 2.

x-2 (x= 2)

$$\begin{array}{rrrr} 1 & 2 & -9 & -18 \\ & 2 & 8 & 2 \\ \hline 1 & 4 & -1 & 20 \end{array} \quad \text{NO.}$$

Now x = - 2.

x+2 (x= -2)

$$\begin{array}{rrrr} 1 & 2 & -9 & -18 \\ & -2 & 0 & 18 \\ \hline 1 & 0 & -9 & 0 \end{array} \quad \text{YES.}$$

So we have:

p(x) = $x^4 + 3x^3 – 7x^2 – 27x – 18 = (x+1)(x+2)(x^2 – 9)$

p(x) = (x + 1)(x + 2)(x + 3)(x – 3)

The solutions are x = – 1, – 2, 3, and – 3.

Solve $x^3 + 5x^2 + 12x + 8 = 0$

Start with 1.

x-1 (x= 1)

$$\begin{array}{rrrrl} 1 & 5 & 12 & 8 & \\ & 1 & 6 & 18 & \\ \hline 1 & 6 & 18 & 26 & \text{NO.} \end{array}$$

x = -1

x+1 (x= -1)

$$\begin{array}{rrrrl} 1 & 5 & 12 & 8 & \\ & -1 & -4 & -8 & \\ \hline 1 & 4 & 8 & 0 & \text{YES.} \end{array}$$

We get (x + 1)(x^2 + 4x + 8)

Does $x^2 + 4x + 8$ factor? No – quadratic formula

We have x = $\dfrac{-4 \pm \sqrt{16-32}}{2}$

x = $\dfrac{-4 \pm \sqrt{-16}}{2}$

x = - 2 + 2i, - 2 – 2i

The solution is x = – 1, -2 + 2i, -2 – 2i.

Rational Zeros Theorem

For a polynomial with leading coefficient of an and constant term a_0 the possible zeros will have the form $\frac{p}{q}$ where p is a factor of a_0 and q is a factor of a_n.

Example:

To factor $4x^4 + 8x^3 – 7x^2 – 27x – 15$ we have:

p: +/- 1, 3, 5

q: +/- 1, 2, 4

This gives us the following possible zeros: +/- 1, $\frac{1}{2}$, $\frac{1}{4}$, 3, 3/2, $\frac{3}{4}$, 5, 5/2, 5/4

The Remainder Theorem: When dividing a polynomial p(x) by the factor (x – k) the remainder is equal to p(k).

Said another way, when you test a value k using synthetic division, if you get zero as a remainder, then k is a zero and p(k) = 0. If you don't get zero as a remainder, then k is not a zero and the remainder is p(k).

EXAMPLE: Use synthetic division to find p(4) if p(x) = $2x^3 + 5x^2 – 27x + 8$

4 2 5 –27 8 8 52 100 2 13 25 108
So we have p(4) = 108.

18.3 Graphing Polynomials

Polynomials are special functions, and their graphs share basic properties we can use to produce sketches of their graphs.

Every graph of a polynomial takes one of four basic shapes, and we can know which one simply by looking at the leading term, the first term, of that polynomial.

NOTE: The terms of a polynomials should always occur in the descending order of their exponents. The first term should have the highest exponent.

The first term has the form $a_n x^n$.

The shape of the graph of a polynomial are determined by the sign of an and the whether the exponent is even or odd.

The Leading Term Test – The Four Polynomial Shapes

1. an is positive and n is even.

2. an is positive and n is odd.

3. an is negative and n is even.

4. an is negative and n is odd.

If you think about it, the odd or even of the exponent really determines the shape, and the sign of an simply points that shape in one of two possible directions.

Why does the leading term alone determine the shape of the graph? Because as x gets larger and larger, all the other terms become insignificant compared to the leading term.

Also, as the polynomials gets higher and higher exponents, and more terms, the graph can change direction more and more times.

Second order – quadratic. Changes direction once.

Third order – cubic. Can change direction twice.

Fourth order – Can change direction three times.

These direction changes give rise to what I call the "Squiggle"

To graph a polynomial, FIRST – look at the leading term. Which of the four shapes do we have? What is the order (highest exponent)? What kind of squiggle might we be looking at?

SECOND – Just cause it's so easy, plot the point f(0). Just put in x = 0. Every term is going to disappear except the last one. Plot that point on your graph.

THIRD – find the zeros of the polynomial and plot them. THIS IS WHERE THE WORK IS. Once you have the zeros, PLOT THEM! You know they are on the x-axis. THAT'S WHAT IT MEANS TO BE A ZERO!!

At this point, you have done the heavy lifting. You've done the tedious part and can now use your understanding to sketch a decent graph now that you know the basic shape, zeros on the x-axis, the point on the y-axis at x=0.

DRAWING

To graph a higher order polynomial:

1. Use leading term test to determine the basic shape.

2. Plot p(0) on the y-axis.

3. Factor p(x) and plot the zeros on the x-axis

4. Connect the dots to fit the basic shape.

5. Clarify the graph by plotting additional points as desired.

Graph: $p(x) = x^3 – 3x^2 – 10x + 24$

1. Positive odd

2. P(0) = 24

3. Zeroes are x = 2, 4, − 3

4. Draw

Graph: $p(x) = –x^3 – 3x^2 + x + 3$

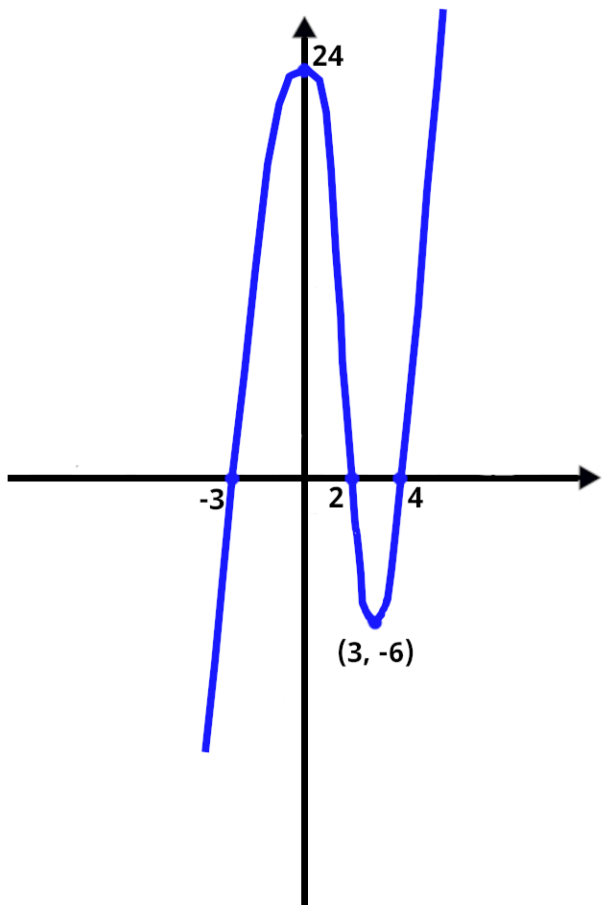

Figure 18.2: The graph of $p(x) = x^3 - 3x^2 - 10x\ +\ 24$

1. Negative odd

2. P(0) = 3

3. Zeros are 1, – 1, and – 3.

Graph $p(x)\ =\ x^4\ +\ 3x^3 - 7x^2 - 27x - 18$

1. Leading term test - Positive Even

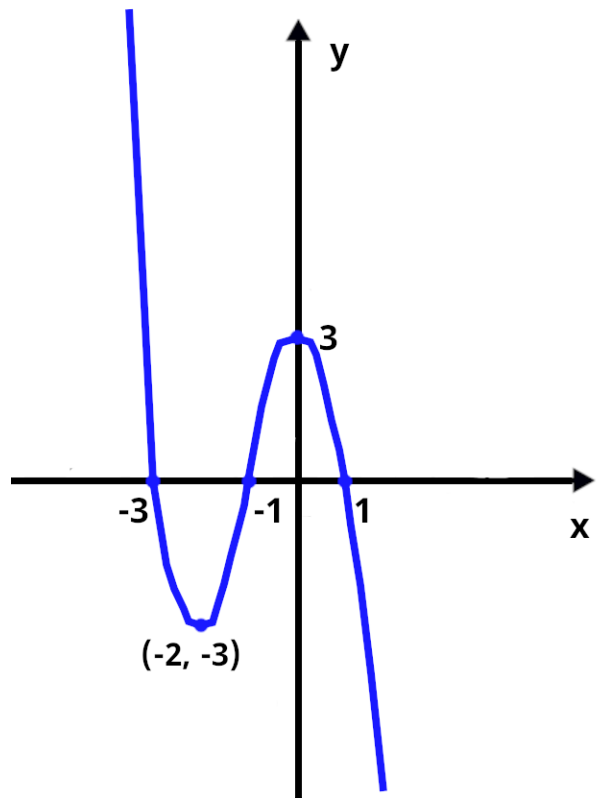

Figure 18.3: The graph of $p(x) = -\ x^3 - 3x^2\ +\ x\ +\ 3$

2. We have p(0) = – 18

3. The zeros are –3, –2, –1, and 3.

4. Connect the dots matching the required shape.

Graphing polynomials of order higher than three requires additional analysis of the behavior in the interval where the "squiggle" occurs. Depending on the precision desired, one evaluates the value of the function at points of interest.

In this example, we have zeroes at -3, -2, -1, and 3. As a result, points of interest are those between x = -3 and x = 3. We know that the graph turns around between -3 and -2, and between -2 and -1, as a result, the points halfway between them are of interest.

We would like to know the value of p(x) for x $= -2\frac{1}{2},\ x = -1\frac{1}{2},\ and\ x = 2.$

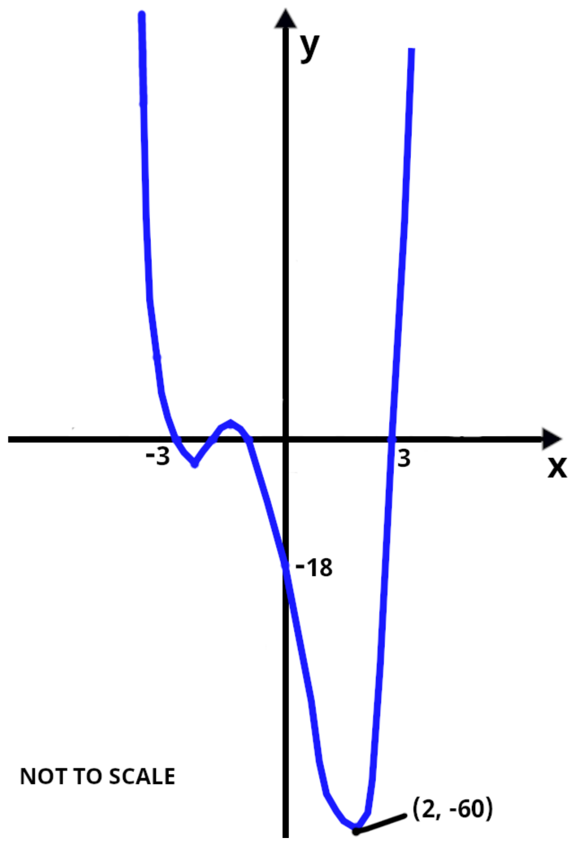

Figure 18.4: The graph of $p(x)\ =\ x^4\ +\ 3x^3 - 7x^2 - 27x - 18$

Evaluate p(x) at critical points.

p $(-\frac{5}{2}) = -\frac{33}{16}$

p $(-\frac{3}{2}) = \frac{27}{16}$

p(1) = – 48

p(2) = – 60

Example: Graph $p(x) = x^3–3x^2–4x + 12$

1. Leading term test – positive odd

2. P(0) = 12

3. The zeros are – 2, 2, and 3.

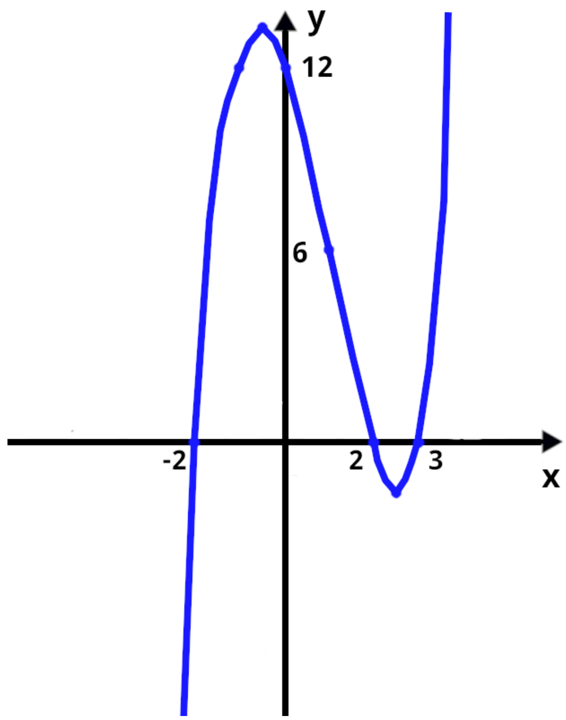

Figure 18.5: The graph of $p(x) = x^3 - 3x^2 - 4x + 12$

We should at least find the value at x = 5/2.

The value of p(5/2) = – 9/8

The value of p(1) = 6

The value of p(– 1) = 12, so we can see that it doesn't turn around at 0.

It probably turns around very close to $\frac{-1}{2}$, and if we wanted, we could calculate the value at x = $\frac{-1}{2}$.

Zeros with multiplicity greater than one:

Graph p(x) = $x^3 + 11x^2 + 35x + 25$

1. Leading term test – positive odd

2. F(0) = 25.

3. Factor and we get p(x) = $(x + 1)(x + 5)^2$

Zeros are – 1 and – 5, but the (x + 5) factor is squared. This zero has multiplicity of 2.

If multiplicity is odd, the graph passes THROUGH the zero.

If multiplicity is even, the graph touches the x-axis and turns around. This is also called a tangent point.

Graph p(x) = $-x^4 + 3x^3 + 3x^2 – 11x + 6$

1. Negative even

2. P(0) = 6

3. Zeros are 1 (multiplicity 2), – 2, and 3.

Suppose we are asked to graph a quadratic?

Graph f(x) = $3x^2 – 18x + 32$.

While the procedure for graphing higher order polynomials can apply to those of second order (quadratics), putting the function into vertex form is more efficient, in particular because vertex form provides a precise vertex for the parabola.

We get 3 (x^2 – 6x) + 32

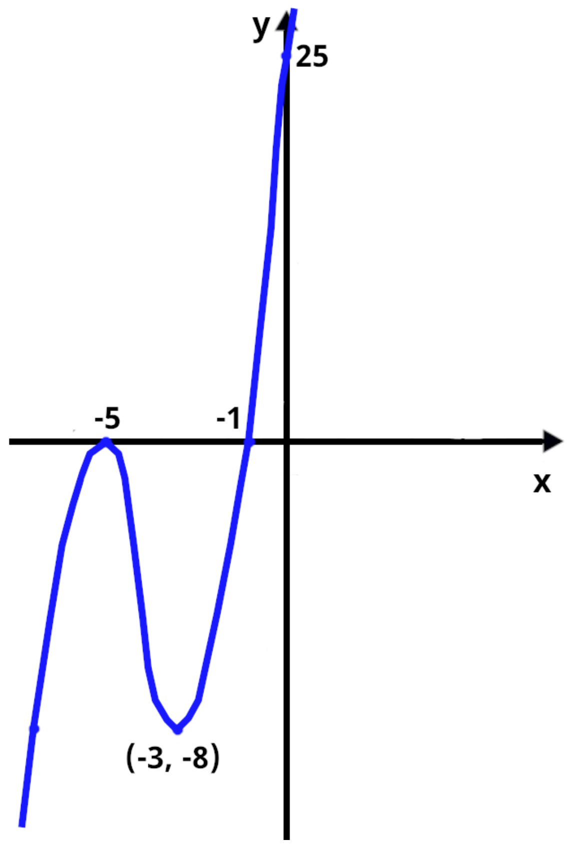

Figure 18.6: The graph of $p(x) = x^3 \ + \ 11x^2 + \ 35x + 25$

3 (x^2 − 6x + 9) - 27 + 32

3 (x − 3) ^{2} − 27 + 32

3 (x − 3) ^{2} + 5

We have a vertex at (3,5) and the shape of $3x^2$.
When producing graphs in an algebra course, the level of precision and the amount of labeling of points depends on the course objectives and the instructor. In this course we emphasize the understanding of the graph.

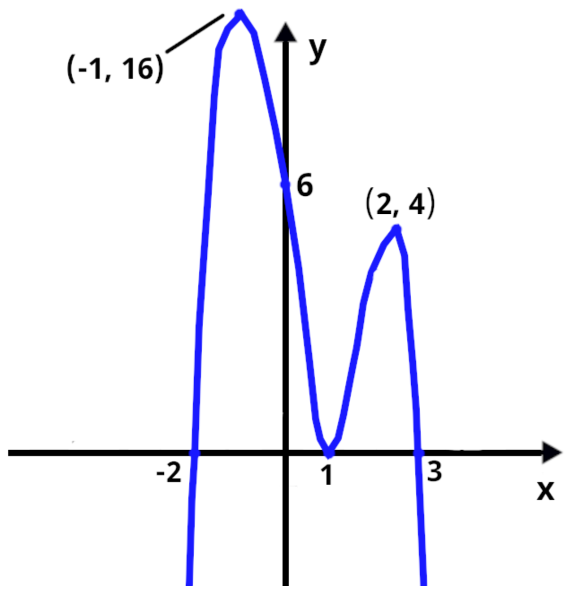

Figure 18.7: The graph of $p(x) \; = x^4 \; + \; 3x^3 - 3x^2 - 11x + \; 6$

For a parabola, specifying the vertex and the shape ax^2 (including the value of a) provides complete information. For cubic polynomials, plotting the intercepts (the y-intercept and the zeros) and sketching the curve that connects them is usually sufficient. If desired, one can evaluate the function at additional points to clarify when the slope of the curve changes sign (when it "turns around").

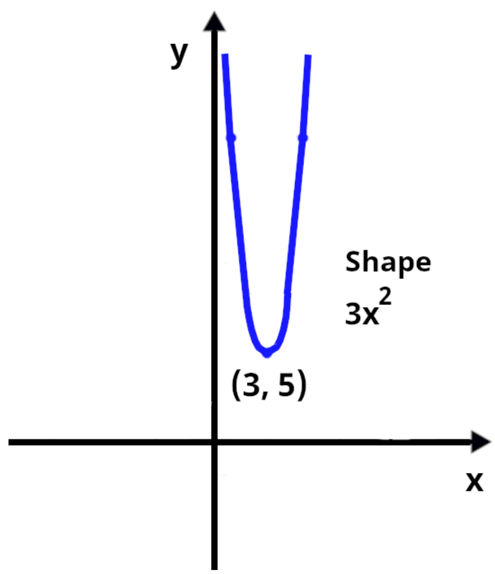

Figure 18.8: The graph of $p(x) = 3x^2 - 18x + 25$