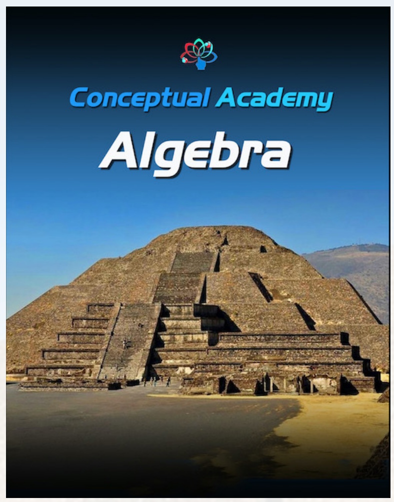
Conceptual Math
Algebra I

Chapter 19: Exponential and Logarithmic Functions



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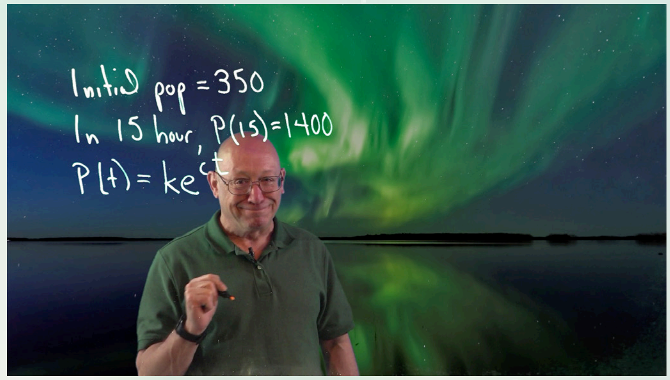

Chapter 19

Exponential and Logarithmic Functions

19.1 Exponential Functions

An exponential function has the form $f(x) = ka^x$ where k and a are constants with $a > 0$ and $a \neq 1$. Note that this is very different from a polynomial with terms having a variable x raised to an integer exponent.

Here the variable is in the exponent. For our purposes, we will set k = 1 and consider exponential functions of the form $f(x) = a^x$. The number raised to a power is known as the base, and of course x is the exponent.

Exponential functions take one of two forms, exponential growth or exponential decay.

When a > 1, f(x) = a^x results in exponential growth. Letting a = 2 or $f(x) = 2^x$ and plotting points from x = -3 to x = 3, we get the table:

x	2^x
- 3	$\frac{1}{8}$
- 2	$\frac{1}{4}$
- 1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

The exercises have you do this for x = 3 and x = 4, but note that if we change the value of a to any value greater than one, the fundamental shape of the graph does not change. For any value of a > 1, we obtain the following graph.

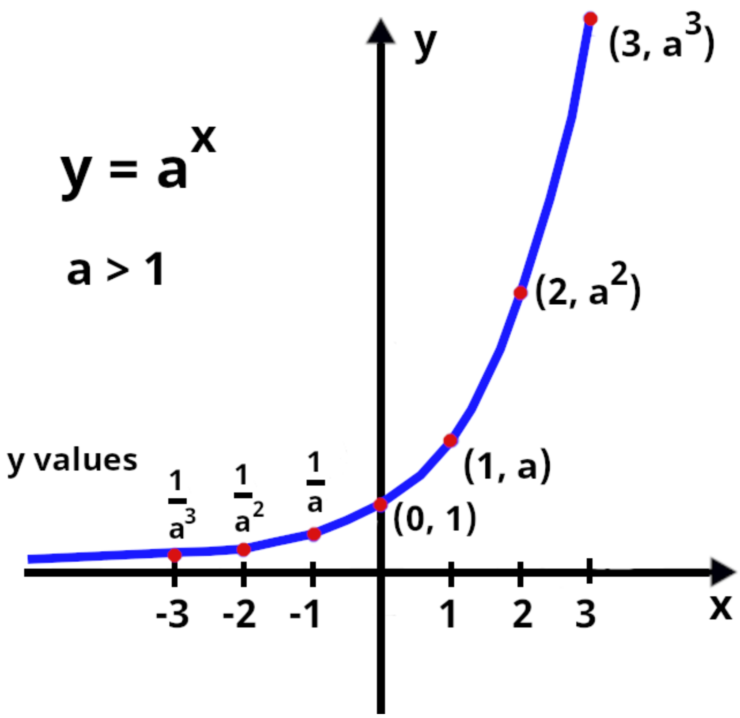

Figure 19.1: The graph of $y = a^x$ *with* $a > 1$

At x = 0, the value of a^x is 1 regardless of the value of a, but as x increases the function rises dramatically in what we know as "exponential growth." In the negative direction the graph approaches zero as the denominator approaches infinity.

Exponential functions can also take the form f(x) = a^{-x}. This flips the graph horizontally about the y-axis. In this case, we have exponential decay. The value of the function approaches zero as x increases. Producing the table for f(x) = 2^{-x}

x	2^{-x}
- 3	8
- 2	4
- 1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

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For ANY real number a $\neq$ 0, for both f(x) = a^x and f(x) = a^{-x} the graph will pass through the point (0,1).

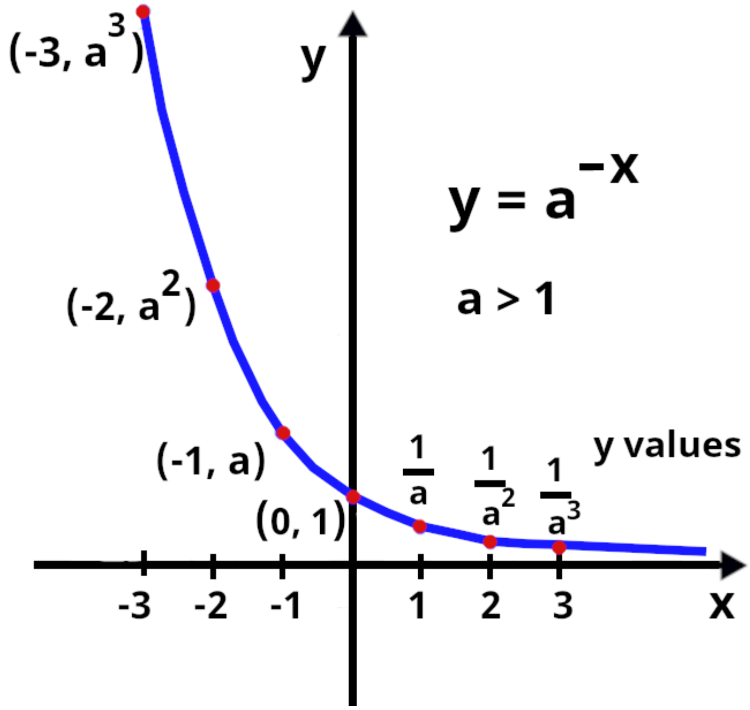

Figure 19.2: The graph of $y = a^{-x}$ *with* $a\ >\ 1$

The function f(x) = a^x is defined for 0 < a < 1, but as we will see that this produces the same function as the base $\dfrac{1}{a}$ with the opposite exponent.

For example, setting $a = \frac{1}{2}$, we get the same tables we saw with a = 2, but in reverse order.
For a = $\frac{1}{2}$ we have

x	$(\frac{1}{2})^{-x}$
- 3	$\frac{1}{8}$
- 2	$\frac{1}{4}$
- 1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

If we consider the four situations: 1) f(x) = a^x, 2) f(x) = a^{-x}, 3) f(x) = $(\dfrac{1}{2})^x$, 4) f(x) = $(\dfrac{1}{2})^{-x}$, we see that our four situations are in fact just two. This follows from the fact that $\frac{1}{2} = 2^{-1}$.

Notice that $(\frac{1}{2})^x = (2^{-1})^x = 2^{-x}$

And $(\frac{1}{2})^{-x} = (2^{-1})^{-x} = 2^x$

Said in a more general way, replacing the base a with $\frac{1}{a}$ has the same result as changing the sign of the exponent, or

$(\frac{1}{a})^x = (a^{-1})^x = a^{-x}$

$(\frac{1}{a})^{-x} = (a^{-1})^{-x} = a^x$

This is true for any base of an exponential function.

As we learned in geometry, The nature of reality includes special numbers that describe certain relationships. The number $\pi = 3.14259\ldots$ provides the circumference of a circle in terms of its diameter (as well as its area and other items). Exponential growth and decay in physics, chemistry, and other areas also operate with a special number.

We designate this number with a lower case e. Like π, e is an irrational number with an infinite number of decimal points that do not repeat.

The number is e = 2.7182818. . . .. When it is the base of an exponential function, it can describe numerous phenomena in the natural world.

The function $y = e^x$ describes the exponential growth of populations such as bacteria in a petri dish. The function $y = e^{-x}$ describes exponential decay, such as what occurs with radioactive isoptopes.

19.1.1 Exponential Growth and Decay

In most practical applications involving exponential growth or decay, the function involves time, and we use t instead of x, and we will only address positive values for time. The function $y = Ce^{kt}$ models the exponential growth and decay where t = 0 represents an initial condition.

The values of c and k are constant. If k > 0, we have growth. If k < 0, we have decay. C is the initial value of the function when t = 0.

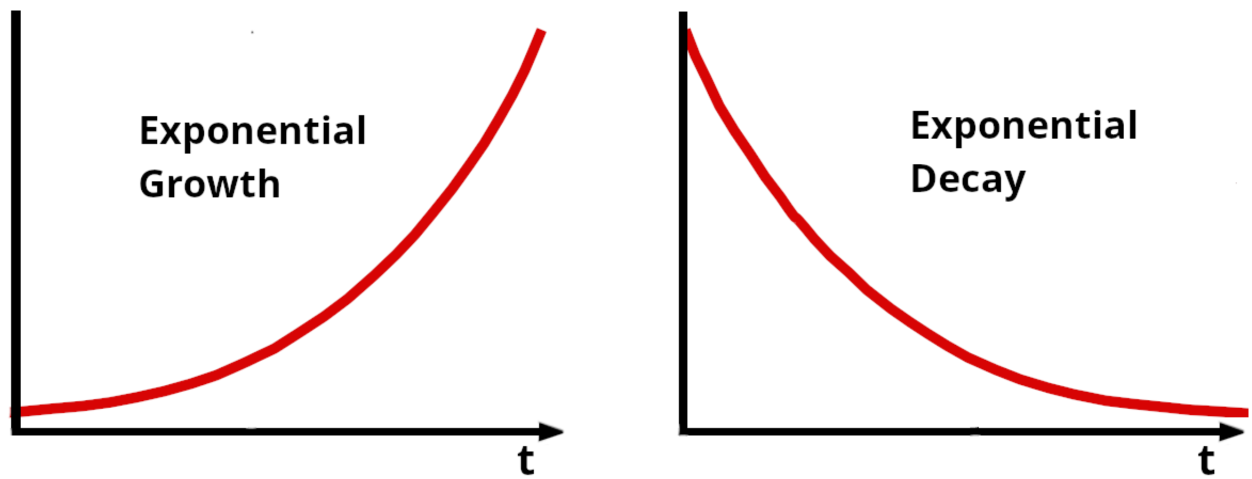

Figure 19.3: Exponential Growth and Decay

EXAMPLE:

Studies suggest that mobile data traffic in exabytes / month is given by the function $A(t) = 3.7e^{0.44t}$ where t is in years since 2015.

a. What is the traffic in 2019? b. What traffic do we predict for 2025?

We have t = 4 for 2019.

$A(4) = 3.7e^{0.44(4)} = 3.7e^{1.76} = (3.7)(5.81) = 21.5$

We have t = 10 for 2025.

$A(10) = 3.7e^{0.44(10)} = 3.7e^{4.4} = (3.7)(81.45) = 301.4$

EXAMPLE:

Strontium 90 is a powerful radioactive isotope. Its level of radioactivity is given by $A(t) = A_0^{-0.024t}$ where A_0 is the initial radiation and t is in years. The Chernobyl disaster released radiation into the environment such that the Strontium 90 level was 8 Ebq. If the safe level of radiation is 0.75 Ebq, is the environment safe after 20 years? 50 years? 100 years?

At 20 years we have $A(20) = 8e^{-0.024(20)} = 8e^{-0.619} = 4.95\ Ebq$.
NOT SAFE.

At 50 years we have $A(50) = 8e^{-0.024(50)} = 8e^{-1.2} = 2.4\ Ebq$.
NOT SAFE.

At 100 years we have $A(100) = 8e^{-0.024(100)} = 8e^{-2.4} = 0.726\ Ebq$.
SAFE (but close to limit).

19.1.2 Inverses of Functions Revisited

The inverse of a function is another function that performs the opposite operation on the input variable.

Examples:

The inverse / opposite of f(x) = x + 5 is g(x) = x – 5

The inverse / opposite of f(x) = 5x is g(x) = x / 5

The inverse / opposite of f(x) = x – 8 is g(x) = x + 8

The inverse / opposite of f(x) = x/15 is g(x) = 15x

Said another way, an inverse of a function "reverses" what the function does. Rembmer that the inverse of f(x) is denoted $f^{-1}(x)$.

As we discussed in section 14.6, applying both the function and its inverse (the composition of f(x) and $f^{-1}(x)$) produces no change to the input.

What is the inverse of an exponential function?

19.2 Logarithmic Functions

Given the function $f(x) = 2^x$, what is its inverse?

What is the number x such that $2^x = 64$?

What is the number x such that $3^x = 27$?

The concept that answers the question of what exponent raises a particular base to a particular value is the logarithm.

A logarithm is an exponent.

If $2^x = 64$, then x $= .log_2 64$.

Definition: $log_a M =$ b MEANS that $a^b =$ M.

Confirm for yourself that:

$log_2 8 = 3$

$log_5 25 = 2$

$log_4 64 = 3$

$log_3 81 = 3$

$log_2 128 = 7$

$log_7 49 = 2$

Now, consider $a^{log_a(x)}$. What is this saying "in English"? We are raising the number a to what power? What does $log_a(x)$ mean by definition?

Question: What color does Bob paint a car if after painting the car, the car is red?

A possibly humorous side note: Many years ago in certain circles the question, "Who is buried in Grant's tomb?" was used to poke fun at painfully obvious questions that contain their own answers.

Students new to logarithms may not consider the result of $a^{log_a(x)}$ obvious at all.

BUT IT IS.

LOG	MATH QUESTION	WORD QUESTION	RESULT
$log_a 1$	$a^? = 1$	a raised to what is 1?	0
$log_a a$	$a^? =$ a	a raised to what is a?	1
$log_a a^2$	$a^? = a^2$	a raised to what is a^2?	2
$log_a a^x$	$a^? = a^x$	a raised to what is a^x?	x

The logarithm with base e is called the NATURAL logarithm and is denoted by ln(x).

ln (32) is the same as $log_e 32$

log (32) is the same as $log_{10} 32$

Most calculators have both the log and ln functions to compute the desired logarithm of either base. Calculators almost never have keys to compute the logarithm in a different base. Advanced calculators may have logarithm functions that allow one to specify a base other than 10 or e, but these are not necessary.

A simple formula allows one to calculate the logarithm for any base.

The change of base formula:

$$log_b\ M = \frac{log_a\ M}{log_a\ b}$$

Since the number a can be any logarithm base, one is free to choose 10 or e.

Example: Use a calculator to find $log_7\ 200$.

Using base 10, we have $log_7\ 200 = \dfrac{log\ 200}{log\ 7}$

which gives us $\dfrac{2.3010}{0.8451} = 2.7228$.

To check this result, use the calculator and raise 7 to the power 2.7228. We get 199.984 due to rounding to four decimal places.

This formula allows you to calculate any logarithm with any base using the established log functions for base 10 and base e in your calculator.

PROPERTIES OF LOGARITHMS

1. $log_b 1 = 0$ True by definition

2. $log_b b = 1$ True by definition

3. $log_b b^x$= x True by definition

4. $log_b MN = log_b$ M + log_b N Product Rule

5. $log_b \dfrac{M}{N} = log_b$ M – log_b N Quotient Rule

6. $log_b\ M^P$ = P log_bM Power Rule

7. $log_b\ M$ = logb N if and only if M = N. Equality

Applying the properties of logarithms to simplify or manipulate expressions is the easiest way to become adept at using them.

Examples:

Write log (xy) as a sum of logarithms. *Solution:* log x + log y.

Write log $(xy)^4$ as a sum of logarithms. *Solution:* From the power rule we have 4 log (xy) = 4 (log x + log y) = 4 log x + 4 log y.

Given that log_b M = 2, log_b N = 7, and log_b Q = 3, evaluate $log_b(\frac{M^2N}{Q^3})$.

Solution: Using the quotient rule we can write this as log_b (M^2N) - log_b Q^3.

Using the product rule: log_b $M^2 + log_b$ N - log_b Q^3

Using the power rule: 2 log_b M + log_b N - 3 log_b Q

Plugging in the values provided: 2(2) + 7 - 3(3) = 4 + 7 - 9 = 2.

Students new to logarithms (or not new to logarithms) should be aware of some common errors that people make when learning the properties of logarithms.

WARNINGS	
The log of a sum is NOT the sum of the logs.	log (a + b) $\neq$ log a + log b
-	
The log of a product is NOT the product of the logs.	log (ab) $\neq$ (log a) (log b)
-	
The log of quotient is NOT the quotient of the logs	log $(\frac{a}{b}) \neq \frac{log\ a}{log\ b}$
-	
A log raised to a power is not the power rule.	$(log\ a)^P \neq$ P (log a)

19.3 Exponential and Logarithmic Equations

19.3.1 Exponential Equations

When the exponents in an equation contain variables, we say that it is an exponential equation.

$5^x = 125$ and $3^{2x} = 81$ are examples of exponential equation.

In some cases, like the two equations above, we can write both sides of the equation as the same number raised to a power.

Since $125 = 5^3$, we can write the first equation as $5^x = 5^3$.

Recall from the properties of exponents.

For any a > 0, a ≠ 1, we have $a^x = a^y \leftrightarrow x = y$.

Using this, we can set the exponents equal to each other, giving us x = 3, solving the equation.

Similarly, since $81 = 3^4$, we can write the second equation as $3^{2x} = 3^4$.

Setting the exponents equal: 2x = 4 or x = 2.

When possible, write the equation as the same base raised to the appropriate power. The exponents must be equal, producing an equation we know how to solve.

Example: $2^{x+5} = 8^{2x}$

Since $8 = 2^3$, we can replace 8 with 2^3 to get

$2^{x+5} = (2^3)^{2x}$

$2^{x+5} = 2^{6x}$

Now we can set the exponents equal to each other

x + 5 = 6x

5 = 5x

x = 1.

For exponential equations where it is not possible to write the two sides as the same base raised to given powers, it is necessary to introduce logarithms and use the properties of logarithms to produce an equation that is easy to solve.

First, let's recall an important fact about logarithms:

For any M > 0, N > 0, and a > 0, a ≠ 1, we have log_a M = log_a N ↔ M = N.

Some find this property easier to remember as "If their logs are equal, the numbers are equal" with the understanding that the logarithms must have the same base. IMPORTANT – this property is most often applied by using it to take the log of each side of the equation, knowing that the resulting equation is also true.

Example: $5^x = 70$

Solution:

1. Take the log of both sides.

2. Use the properties of logarithms to produce another equation.

We have $\log 5^x = \log 70$

Power rule x log 5 = log 70

Divide by log 5 $x = \frac{log\ 70}{log\ 5}$

If interested, using a calculator we could compute the value of x. In many courses leaving the answer as an operation on specific log values is preferred.

Example: $2^{x+5} = 7^x$

Take log of both sides $\quad \log 2^{x+5} = \log 7^x$

Power Rule $\quad$ (x+5) log 2 = x log 7

Distribute $\quad$ x log 2 + 5 log 2 = x log 7

Subtract x log 2 $\quad$ 5 log 2 = x log 7 – x log 2

Factor out x $\quad$ 5 log 2 = x (log 7 – log 2)

Divide $\quad \dfrac{5\ log\ 2}{log\ 7 - log\ 2} = x$

Of course we could write $x = \dfrac{5\ log\ 2}{log\ 7 - log\ 2}$.

As an algebra task, we usually leave the answer in this form. In an engineering course or when solving an application problem, we can use a calculator and compute this value to get x = 2.7665.

19.3.2 Logarithmic Equations

Logarithmic equations are equations that contain variables in logarithmic expressions, such as log x = 3 or ln x + ln (x – 2) = 5. Similar to exponential equations, the solution involves using the properties logarithms to put the equation into a form that leads to the solution. In this case, we use the properties to write the equation in terms of a single logarithm. If the equation already consists of a single logarithm, then this initial step is already complete.

EXAMPLE: $log_4 x = 3$

This is the simplest form of logarithmic equation where the definition of the logarithm provides the information necessary to solve for x. By definition, we know that 4^3 = x, telling us that x = 64.

EXAMPLE: log (x + 6) – log(x – 2) = log 5

The left hand side is a difference of logs. From the quotient rule we can write this as a single logarithm.

$\log \dfrac{x+6}{x-2} = \log 5$

If log a = log b, then we have a = b, allowing us to produce the equation:

$\dfrac{x+6}{x-2} = 5$

x + 6 = 5 (x – 2)

x + 6 = 5x – 10

x + 16 = 5x

16 = 4x

x = 4.

When solving logarithmic equations, it is necessary to check the solutions in the original equation to insure that the value(s) found are defined for the logarithms involved.

Logarithms are not defined for zero or negative numbers, and in the above equation we take the log of x+6 and x-2. Are either of these negative when x = 4? No - x = 4 is a solution. If our answer had been 1, we would have rejected it and concluded that the equation has no solution.

EXAMPLE: log x + log (x+15) = 3

Here we have the sum ot two logs, so by the product rule we get:

log (x)(x+15) = 3.

By the definition of logarithms, we know that (x)(x+15) must equal 10^3 = 1000..

So we have x^2 + 15x = 1000

Or x^2 + 15x – 1000 = 0

We can solve this quadratic equation by factoring or by using the quadratic formula. Either method yields the solutions of 25 and – 40. Consider these values in the original equation. We cannot use the value of x = – 40 . Our solution is x = 25.

Suggestion: With your calculator, confirm that log 25 + log 40 = 3.